DYNAMICS AND THE GODBILLON-VEY CLASSES: A HISTORY AND SURVEY

STEVEN HURDER

ABSTRACT. We survey thirty years of study of the relations between dynamics and the Godbillon-Vey invariant of codimension one foliations. We include a section on open problems.

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1. A simple definition

In 1971, C. Godbillon and J. Vey introduced the invariant of foliations named after them. Previously, the study of foliations was considered either as an area of topology, viewing a foliation is a generalized fibration structure on a manifold (cf. [76]), or as an area within differential equations, whose key results concerned recurrence and limit sets (cf. [97]). With the advent of the Godbillon-Vey class and its generalization to the other secondary classes, the field obtained a unique focus, characterized by the interplay of geometry, topology, dynamical systems, and eventually ergodic theory. The study of the Godbillon-Vey invariant for codimension one foliations, in particular, illustrates the breadth of the field of foliations, and it is still a developing subject. We discuss here the path of ideas and results from "a simple definition", to our current understanding of how the Godbillon-Vey class depends on the geometry and dynamics of a foliation, concluding with a selection of open problems. This work updates and expands on parts of the Seminaire Bourbaki "Sur l'invariant de Godbillon-Vey" by Ghys [31], and the problem session from the 1992 foliations meeting in Rio de Janeiro [72]. The bibliography attempts to be comprehensive and up-to-date.

Let M be a Riemannian C^∞–manifold with a codimension–one foliation $\mathcal{F}$ which is transversally C^2, and with C^∞ leaves. We assume that the normal bundle to $\mathcal{F}$ is oriented, so there is a 1-form ω on M whose kernel defines the leaf tangent bundle $T\mathcal{F}$. The Frobenius Theorem implies there is a 1–form η on M such that $d\omega = \eta \wedge \omega$. Godbillon and Vey [41] observed that 3-form $\eta \wedge d\eta$ is closed, and its cohomology class $GV(\mathcal{F}) = [\eta \wedge d\eta] \in H^3(M)$ is an invariant under diffeomorphism and foliated concordance [77].

The simplicity of the definition of the Godbillon-Vey invariant $GV(\mathcal{F}) = [\eta \wedge d\eta]$ gives little indication of its geometric or dynamical meaning. The 1-form η entering in the definition is sometimes called the *modular form* for $\mathcal{F}$. It has long been known that the restriction of η to leaves of $\mathcal{F}$ are closed forms, and for a C^1-foliation there is a well-defined leafwise cohomology class $[\eta] \in H^1(M, \mathcal{F})$, called the *Reeb class*, and the integral of η along leafwise curves measures the transverse holonomy expansion [95] (see also [43, 96] and § 2, [31]) . The idea can even be traced back to the work of Poincaré [93]. So $[\eta]$ is a cohomology invariant of $\mathcal{F}$ which measures the transverse expansion.

The study of dynamics of flows and foliations have both been profoundly influenced by the special examples derived from Anosov flows on compact manifolds [1, 91, 92], especially the Anosov flows which are the geodesic flow of a compact manifold with constant negative curvature. For example, let M denote the unit tangent bundle to a metric of constant curvature on a closed orientable surface Σ_g of genus $g > 1$, and $\mathcal{F}$ the weak-stable foliation of the geodesic flow on M. The Roussarie calculation, included in [41], evaluated $GV(\mathcal{F})$ on the fundamental class, $\langle GV(\mathcal{F}), [M]\rangle = \pm 4\pi(1-g)$, giving the first example of a codimension one foliation with non-vanishing $GV(\mathcal{F})$. Thurston's celebrated construction [106] of a family of smooth foliations $\{\mathcal{F}_t \mid t > 0\}$ on the 3-sphere, for which $\langle GV(\mathcal{F}_t), [M]\rangle = t$, is constructed from the weak stable foliation starting with a punctured surface. The leaves of the weak stable foliation of an Anosov flow have exponential growth and are transversally expansive. Moussu and Pelletier [86] and Sullivan (in [102]) asked whether this is always true - if $GV(\mathcal{F}) \neq 0$ must $\mathcal{F}$ have leaves of exponential growth? This conjecture was supported by all of the examples. Moreover, in Thurston's examples the Godbillon-Vey class assumed a continuous range of values, suggesting that a geometric interpretation of $GV(\mathcal{F})$ should involve dynamical information such as "entropy".

Thurston's note [106] also contained a geometric interpretation of the Godbillon-Vey invariant for 3-manifolds as measuring "helical wobble". Reinhart and Wood [98] gave a formula expressing the the Godbillon-Vey class of a foliation of a Riemannian 3-manifold in terms of the curvatures of the leaves and normal bundle, which can be interpreted as the helical wobble. (See [31] for a nice discussion on this point.) As pointed out by Langevin, the Reinhart-Wood formula is suggestive of an integral geometry interpretation of the Godbillon-Vey invariant and hints at possible dynamical connections [3, 73]. Sullivan [105] gave another interpretation of the Godbillon-Vey invariant as the "second derivative" of the intersection of certain foliation currents. Sullivan's calculation is also an interpretation of helical wobble, but more in the spirit of dynamical systems.

2. Structure theory

During the period 1976–1982, there were a succession of works by various authors which proved the Moussu-Pelletier and Sullivan conjecture for increasingly general classes of foliations. The approach to the conjecture was to prove a "vanishing theorem", where the leaves of $\mathcal{F}$ are assumed to have a dynamical property which implies subexponential growth, and then show this forces $GV(\mathcal{F}) = 0$.

The first result was by Herman [53] in 1976, who showed that $GV(\mathcal{F}) = 0$ if $\mathcal{F}$ is the suspension of an action of $\mathbb{Z}^2$ on the circle. This was extended soon after by Wallet [121], so that one knew that $GV(\mathcal{F}) = 0$ for a codimension one C^2-foliation $\mathcal{F}$ transverse to a circle bundle over a compact Riemann surface, if no leaf of $\mathcal{F}$ has holonomy. Wallet's result was in turn extended by Morita and Tsuboi [84] in 1978, who showed that $GV(\mathcal{F}) = 0$ if $\mathcal{F}$ is a foliation without holonomy on any compact manifold. This result introduced C^2-dynamics and Sacksteder's Theorem into proofs of the "vanishing theorem" for the first time, for a corollary of the Sacksteder theorem is that a foliation without holonomy is defined by a transverse invariant measure. The vanishing of $GV(\mathcal{F})$ is a consequence of constructing a sequence of smooth transverse 1-forms approximating this measure.

A foliation $\mathcal{F}$ is *almost without holonomy* if the only leaves with holonomy are compact. The simplest example of such is the Reeb foliation, though in general these foliations can have very complicated topology, as they are depth one with the non-compact leaves spiraling in on the compact leaves. Their study has a long history [19, 44, 48, 83, 85, 86, 88], with the basic structure theorem for C^2-foliations proving that the holonomy of the compact leaves must be abelian, that their complement is a union of local minimal sets, and there exists a transverse vector field defined in an open neighborhood of the compact leaves which commutes with the holonomy. Mizutani, Morita and Tsuboi [82] and Cantwell and Conlon [8] proved in 1980 that $GV(\mathcal{F}) = 0$ for foliations almost without holonomy. Both proofs combined the structure theory for foliations almost without holonomy with the techniques developed for foliations without holonomy.

Throughout the 1970's, the geometry of codimension one foliations with increasing degrees of complexity was actively studied. This research culminated in three distinct approaches to a general structure theory: Dippolito proved the "octopus decomposition" for C^0-foliations; Nishimori [89] introduced the "SRH" (Staircase, Room, Hall) decomposition; Cantwell and Conlon [9] and Hector [49] proved a Poincaré-Bendixson theorem for the asymptotic behavior of leaves. All of these can be seen as developing the ideas of the almost without holonomy classification theory. The Poincaré-Bendixson theory also provided an approximate correspondence between the growth rates of leaves and their level of complexity [7, 10, 11, 47, 110, 111, 112, 113, 115]. During 1980-81, a succession of authors extended the vanishing theorem for foliations almost without holonomy, to increasingly more general classes [8, 28, 89, 114]. None of these results directly related the growth of leaves to the vanishing of $GV(\mathcal{F})$, and all relied on the structure theory to estimate the form $\eta \wedge d\eta$.

Underlying these succession of works was a new idea, that the Godbillon-Vey invariant could be "localized" to an open saturated subset $U \subset M$, where the special "dynamics and geometry" of $\mathcal{F}|U$ could be used to show $GV(\mathcal{F}|U) = 0$. This idea became more explicit in the works of Cantwell-Conlon [8], Nishimori [89], Tsuchiya [114], and Duminy-Sergiescu [28].

3. Duminy's Theorem

In a brilliant work growing out of the paper [28], G. Duminy [26] introduced the *Godbillon measure* $g_{\mathcal{F}}$ on the Σ-algebra $\mathcal{B}_O(\mathcal{F})$ generated by the open saturated subsets of a foliation $\mathcal{F}$ of codimension one. Duminy's note was also highly original on two other points: The Godbillon measure is one half of the Godbillon-Vey invariant, constructed from the leaf cohomology class $[\eta] \in H^1(M, \mathcal{F})$. The second half, the "Vey class" $[d\eta] \in H^2(M/\mathcal{F})$, was considered as a fixed invariant of $\mathcal{F}$ on which the Godbillon measure could be evaluated to give $GV(\mathcal{F}|U)$ for $U \in \mathcal{B}_O(\mathcal{F})$. The other innovation, in a companion manuscript [27], was an estimation of the Godbillon measure using what are essentially techniques of ergodic theory. Duminy's work lifted the veil on the study of the relation between $GV(\mathcal{F})$ and dynamics.

THEOREM 3.1. [Duminy] *If $\mathcal{F}$ is a codimension one, C^2-foliation of a compact manifold M with non-trivial Godbillon measure $g_{\mathcal{F}}$, then $\mathcal{F}$ must have a hyperbolic resilient leaf, and hence there is an open subset of M consisting of leaves with exponential growth.*

While the original manuscripts of Duminy were widely circulated, they were unfortunately never published. An account of Duminy's method was later published by Cantwell and Conlon [12], who extended the theorem to non-compact manifolds. This extension is not so simple as passing from cohomology on a compact manifold to that on a non-compact manifold, as the key estimates Duminy used required the Poincaré-Bendixson structure theory of C^2-foliations on compact manifolds [9, 49]. Cantwell and Conlon extended in the Poincaré-Bendison theory to open manifolds in [12].

Duminy's method of proof was that, if there is no resilient leaf for $\mathcal{F}$, then by Sacksteder's Theorem [100] there are no exceptional minimal sets. Hence, by the Poincaré-Bendixson theory, all leaves of $\mathcal{F}$ either lie at finite level, or lie in "arbitrarily thin" subsets $U \in \mathcal{B}_O(\mathcal{F})$. The finite level case is analyzed analogously to the almost without holonomy case, while the analysis of the thin sets used the new techniques of [27]. Duminy's new techniques raised new questions:

1. Exactly what are the properties of the "Godbillon measure"?
2. Could the technique of the Godbillon measure also yield an approach to relating the secondary classes for higher codimension foliations to their dynamics?
3. What is the meaning of the calculation in [27] which miraculously showed just what needed to be shown?
4. Does the "Vey class" have a geometric or dynamical meaning comparable to that proved for the Godbillon measure?

Only the first two of these questions have been partially answered to date.

After Duminy's manuscripts first appeared in March 1982, these questions were discussed among Larry Conlon, André Haefliger, James Heitsch, Paul Schweitzer, and the author, who were visiting Princeton in Spring, 1982. Three of the five participants in this mini-program at the Institute for Advanced Study subsequently published papers on Duminy's work!

The first development following on Duminy's work was by Heitsch and Hurder [52]. They extended Duminy's ideas in two directions, one formal and the other more fundamental. First, they showed that for a codimension $q \geq 1$, C^1-foliation, each cohomology class $y_I \in H^*(\mathfrak{gl}(\mathfrak{q}), \mathfrak{SO}(\mathfrak{q}))$ yields an associated *Weil measure*, denoted by $\chi_{\mathcal{F}}(y_I)$, for which the Godbillon measure $g_{\mathcal{F}} = \chi_{\mathcal{F}}(y_1)$ is the simplest one. The classes y_I are those which appear in the E^2-term of the spectral sequence calculation of the cohomology of the truncated Weil algebra $WO(q)$, hence their name. The fact that the Weil measures are defined for C^1-foliations was just an observation based on the definitions. The definition of the "Vey class", and more generally the classes $[c_J]$ corresponding to the Chern forms $c_J(\mathbf{Q})$ of the normal bundle to $\mathcal{F}$, still require a C^2-foliation for their definition.

The more fundamental point of the work [52] was that the measures were defined on the Σ-algebra $\mathcal{B}(\mathcal{F})$ of all measurable, saturated subsets of M, and that the leafwise forms used to define the measures $\chi_{\mathcal{F}}(y_I)$ need only be transversally measurable. It may happen that the only *open* saturated subsets for a foliation are either empty, or M itself, so the extension of the Godbillon measure to the algebra of measurable sets $\mathcal{B}(\mathcal{F})$ widened the application of the ideas. On the other hand, the ability to calculate these new invariants using measurable data had a more important impact. Dynamical hypotheses on a foliation, combined with asymptotic techniques, often yields leafwise smooth, but only transversally measurable data, so that this extension allowed techniques of ergodic theory to be applied in the evaluation of the Godbillon and Weil measures.

For example, the Godbillon measure $g_{\mathcal{F}}$ can be calculated using a transversally measurable, transverse volume form for $\mathcal{F}$. A foliation with all leaves compact is easily shown to have a transversally measurable, closed transverse volume form, hence its Godbillon-Vey classes must vanish. This argument applies in any codimension. Cantwell and Conlon used the modification of Duminy's approach to give a simpler proof of Duminy's Theorem [12, 13]. Hurder [55] showed that for a foliation of arbitrary codimension, if all leaves are compact, then all of the Weil measures must vanish since such a foliation admits a transversally measurable, holonomy invariant transverse Riemannian metric. This was the first "vanishing theorem" for the other secondary classes of foliations, for codimension greater than one.

Duminy's papers [26, 27] were a demarcation in the study of the dynamics of foliations. The fundamental conjecture posed in 1974 was solved, while the reformulation of his ideas transformed the study of the relation between dynamics of foliations and the Godbillon-Vey and other secondary classes, into questions of ergodic theory. The study of foliation dynamics was afterwards motivated by results and techniques of the ergodic theory for smooth maps [103, 71] and group actions [125]. Connections to the theory of ergodic equivalence relations [29] and cocycles [101] became fundamental.

4. Ergodic theory

One of the "often discovered"' facts in ergodic theory is that a diffeomorphism $f\colon M \to M$ of a compact Riemannian manifold M cannot expand volume on an invariant set of positive measure. This basic fact was first formulated by Schmidt [101] in cocycle language as saying that the additive Radon-Nikodyn cocycle $\nu\colon \mathbb{Z} \times M \to \mathbb{R}$ defined by $\nu(n, x) = \log\{|(f^n)'(x)|\}$ has subexponential growth for almost every $x \in M$. The key to the proof is that the group $\mathbb{Z}$ has subexponential growth, while if $\limsup_{n\to\infty} \nu(n, x)/n \geq a > 0$ on a set $E \subset M$ of positive Lebesgue measure, then the volume of the iterates $f^n(E)$ grows exponentially, and there is "no room" for all the volume in the space $\mathbb{Z} \times M$.

On the other hand, if Γ is a group with exponential word growth acting smoothly on M, then the additive Radon-Nikodyn cocycle $\nu\colon \Gamma \times M \to \mathbb{R}$ can have exponential growth. This is exactly what happens, for example, with the action of a surface group on the circle at infinity in the original examples calculated by Roussarie. This balance between the growth of the group and of the Radon-Nikodyn cocycle underlies the next advance in the study of the Godbillon-Vey invariants.

We need a digression into the growth rates of leaves. For a finitely generated group, the limit $gr(\Gamma) = \lim_{n\to\infty} \log\{\#\Gamma_n\}/n$ always exists, where $\Gamma_n = \{\gamma \mid \|\gamma\| \leq n\}$, as Γ is a homogeneous metric space for the word metric. Given a leaf $L \subset M$ of $\mathcal{F}$ for Riemannian manifold M, and $x \in L$, we say L has *subexponential growth* if $\limsup_{R\to\infty} \log\{Vol_L(x, R)\}/R = 0$ where $Vol_L(x, R)$ denotes the volume in the leaf metric of a ball centered at $x \in L$ with leaf radius R. If $\liminf_{R\to\infty} \log\{Vol_L(x, R)\}/R = 0$, we say that the leaf L has *non-exponential growth*, while L

has *exponential growth* if $\liminf_{R\to\infty} \log\{Vol_L(x,R)\}/R > 0$. A foliation can have leaves which have nonexponential growth but not subexponential growth [47].

In 1984, Hurder [58] proved that for the pseudogroup of a codimension $q \geq 1$ foliation of a compact manifold, if almost all orbits have subexponential growth, then the Radon-Nikodyn cocycle has subexponential growth. This gave the first direct relation between the growth of leaves of a foliation and the asymptotic growth rate of the modular class $[\eta]$. However, subexponential growth of the Radon-Nikodyn cocycle does not imply the modular form has any local growth estimates. A cocycle with subexponential growth may still oscillate wildly on small scale, and only when averaged for large "time" does it behave in a subexponential manner.

One consequence of the Heitsch-Hurder reformulation of Duminy's methods was that it is possible to renormalize the transverse volume form using a leafwise smooth, transversally measurable change of scale function. If such a change of scale function can be chosen so that the new transverse volume form has uniform arbitrarily slow growth, then the modular form η is uniformly estimated arbitrarily close to 0, hence the Godbillon measure must vanish. This is just a measurable version of the original idea from Herman [53]! Fortunately, smooth ergodic theory had already solved the renormalization problem. A fundamental technique for the study of non-uniform hyperbolic dynamical systems is the Lyapunov tempering procedure used in Pesin theory [90]. Hurder and Katok [66] extended this tempering procedure from actions of $\mathbb{Z}$ to the context of metric equivalence relations. When applied to a foliation with almost every leaf of subexponential growth, this yields for each $\epsilon > 0$ a transverse volume form which is uniformly ϵ-invariant. Thus, the Godbillon measure must vanish. This was the key idea behind the proof in [58] of the generalization of the Moussu-Pelletier and Sullivan Conjecture to all codimensions:

THEOREM 4.1. *Let $\mathcal{F}$ be a C^1 foliation of codimension $q \geq 1$ such that almost every leaf has subexponential growth rate. Then the Godbillon measure $g_{\mathcal{F}} = 0$. If $\mathcal{F}$ is C^2 then the Godbillon-Vey class $GV(\mathcal{F}) = 0$.*

The theorem implies that if $GV(\mathcal{F}) \neq 0$ then there exists a set of leaves with positive measure that do not have subexponential growth. It is not known if the set of leaves with non-exponential, but not subexponential growth can have positive Lebesgue measure. Alternately, one can ask if the method of proof of the theorem can be extended to leaves with non-exponential growth.

This discussion is focussed on the development of codimension one theory, but we detour for a moment to describe some related issues that are best observed in higher codimensions. The structure theory for codimension one foliations has no counterpart in higher codimensions. Even for a single diffeomorphism of a compact manifold M of dimension greater than one, it is impossible to give a good classification. The best that has been achieved is to restrict to classes of dynamics, and make a structure theory for diffeomorphisms within these restricted classes. Classes of examples include pseudo-Anosov maps of surfaces, Anosov maps of manifolds, affine actions, projective actions, actions with all orbits finite, distal and equicontinuous actions, and so forth. Similar classes of dynamical systems have been defined and studied for group actions on manifolds with some measure of success, and to some extent also for foliations.

An alternate approach to a classification theory for arbitrary diffeomorphisms, or groups of diffeomorphisms, of a compact manifold M is to attempt to classify the derivative cocycle of the action. Associated to a group action $\varphi\colon \Gamma\times M \to M$ is the derivative cocycle $D\varphi\colon \Gamma\times M \to \mathrm{GL}(\mathbb{R}^q)$, where we have chosen some bounded measurable trivialization of the tangent bundle TM. For a codimension q foliation $\mathcal{F}$, there is an analogous cocycle $D\varphi\colon \mathcal{G}_{\mathcal{F}} \to \mathrm{GL}(\mathbb{R}^q)$ where $\mathcal{G}_{\mathcal{F}}$ is the holonomy groupoid of $\mathcal{F}$. Rather than attempt the seemingly impossible step to classify the group actions or foliation dynamics, one studies their associated cocycles up to an appropriate notion of measurable coboundary. This yields a cohomology theory which is the direct generalization of the idea of the modular class $[\eta] \in H^1(M, \mathcal{F})$. The classification of cocycles up to measurable equivalence is a well-studied problem from smooth ergodic theory of group actions, and there are many techniques, and several celebrated theorems.

One invariant associated to a measurable cocycle is the Lyapunov band spectrum which consists of a band of asymptotic (or generalized) eigenvalues for the range of $D\varphi$. For example, these invariants have a prominent role in the study of asymptotic behavior of the solutions of differential equations, especially in the Sacker-Sell theory. Another more delicate invariant consists of the algebraic hull [127] of $D\varphi$, which is the smallest algebraic subgroup $G \subset \mathrm{GL}(\mathbb{R}^q)$ so that $D\varphi$ is cohomologous to a cocycle with values in G.

The existence of an invariant ergodic measure allows deeper analysis of $D\varphi$. For example, if a smooth action admits an ergodic invariant measure and the group Γ is a higher rank lattice, then Zimmer's Cocycle Superrigidity Theorem [125, 79] reduces the classification of $D\varphi$ to a problem in representation theory. There remains the problem of translating information about the cocycle $D\varphi$ into dynamical information for $\mathcal{F}$, which frequently also requires the hypothesis (or construction) of an invariant measure for the action or foliation. For example, see the applications by Zimmer [123, 124, 126] of cocycle superrigidity to the smooth classification of group actions.

In codimension one, classification theory naturally evolved from an understanding of a certain collection of examples or "models". In higher codimensions, there is a far greater collection of models. The modular form η is replaced by the transverse derivative cocycle $D\varphi$, whose behavior is far more complicated. In spite of these additional considerations, there are several results relating vanishing of secondary classes to foliation dynamics and the classification of the derivative cocycle in higher codimensions. Hurder [56] showed in 1984 that for the linear holonomy of a leaf in a C^2-foliation $\mathcal{F}$, if the algebraic hull is not amenable, then $\mathcal{F}$ has leaves of exponential growth. Hurder and Katok [66] showed in 1987 that if a foliation $\mathcal{F}$ defines an amenable equivalence relation, and hence the algebraic hull of the cocycle $D\varphi \colon \mathcal{G}_{\mathcal{F}} \to \mathrm{GL}(\mathbb{R}^q)$ is amenable, then many of its secondary classes of $\mathcal{F}$ vanish. The University of Chicago thesis of Stuck [104] extended these vanishing results. Analysis of the cocycle $D\varphi$, combined with possible additional hypotheses on the transverse geometry of the foliation $\mathcal{F}$, are expected to yield a better understanding of the dynamics of foliations in higher codimensions. For example, the geometric entropy of a foliation can be combined with information derived from $D\varphi$ to produce new results in all codimensions, as we discuss next.

5. Geometric entropy

One of the most fundamental invariants of the dynamics of a diffeomorphism of a compact manifold is its topological entropy. When positive, it implies the orbits of f exhibit an exponential amount of "chaos". When zero, the map f is somehow not typical, and has unusual regularity. For the study of group actions and foliation dynamics, it is natural to look for a corresponding entropy invariant of the system.

There are been several definitions of topological entropy for group actions, motivated by the need to have a definition for lattice dynamics where the group is typically $\mathbb{Z}^n$. These definitions admit extensions to actions of amenable groups, but for non-amenable groups, these definitions all encounter difficulties of one sort or another. The main problem with adapting these definitions for the study of foliations, however, is that they all vanish for C^1-actions whose leaves have growth rate greater than linear, so would vanish for all but the simplest or most pathological foliations!

The introduction by Ghys, Langevin and Walczak [37] of the geometric entropy $h(\mathcal{F})$ for a C^1–foliation $\mathcal{F}$ marked another transformation in the study of foliation dynamics. Their definition is completely natural: $h(\mathcal{F})$ measures the exponential rate of growth for (ϵ, n)-separated sets in the analogue of the Bowen metrics for the holonomy pseudogroup $\mathcal{G}_{\mathcal{F}}$ of $\mathcal{F}$. (Chapter 13 of [6] gives an excellent introduction and discussion of foliation entropy.) Thus, $h(\mathcal{F})$ is a measure of the complexity of the transverse dynamics of $\mathcal{F}$. The precise value of $h(\mathcal{F})$ depends upon a variety of choices, but the property $h(\mathcal{F}) = 0$ or $h(\mathcal{F}) > 0$ is well-defined. A codimension one foliation with a transverse invariant measure has entropy zero, while the entropy of the Roussarie example is positive. The authors established two key relations between $h(\mathcal{F})$ and dynamics:

THEOREM 5.1. (Theorem 6.1, [37]) *If $\mathcal{F}$ is a C^2–foliation of codimension one, then $h(\mathcal{F}) > 0$ if and only if $\mathcal{F}$ has a resilient leaf*

This theorem, combined with Duminy's Theorem, implies that if $GV(\mathcal{F}) \neq 0$ then $h(\mathcal{F}) > 0$.

THEOREM 5.2. (Theorem 5.1, [37]) *If $\mathcal{F}$ is a C^1-foliation of codimension $q \geq 1$ with $h(\mathcal{F}) = 0$, then $\mathcal{F}$ has a non-trivial holonomy invariant transverse measure.*

A number of problems and questions are suggested by these two theorems (see § 7, [37]):

1. Can one show the implication "if $GV(\mathcal{F}) \neq 0$ then $h(\mathcal{F}) > 0$" directly?
2. Show that if some secondary class is non-zero for $q > 1$, then $h(\mathcal{F}) > 0$.
3. Can one define an analog of metric entropy for a foliation, and use it to estimate $h(\mathcal{F})$?
4. What is the relation between $\mathcal{F}$-harmonic measures and $h(\mathcal{F})$?

These problems have been the continued focus of research since the appearance of the paper [37], and influenced research in many areas of the study of codimension one foliations. We will discuss the progress and updated formulations for these questions.

The problem of defining $h(\mathcal{F})$ using harmonic measures remains open. The closest result to obtaining such a relation, and this is only at the level of intuition, is the paper by Ghys [33] which proves key results about random walks on leaves and recurrence along the ends of leaves. This suggests that $\mathcal{F}$-harmonic measures should almost surely be influenced by the separation of leaves as they tend to infinity, as the entropy measures a type of expansion of the leaves, which should influence the properties of the convergence of the heat flow to harmonic measures. As both the geometric entropy and the properties of harmonic measures are natural invariants of a foliation, any relation between them would have great appeal and most likely be fundamental.

Work on the first three problems will be discussed in relation to the paper [59] by Hurder, which outlined program for the study of $h(\mathcal{F})$ using the foliation geodesic flow. A foliation with smooth leaves has a leafwise geodesic flow, defined on the unit tangent bundle $V = T_1\mathcal{F}$ to the leaves. The foliation $\mathcal{F}$ on M defines a foliation $\hat{\mathcal{F}}$ on V whose leaves cover those of $\mathcal{F}$. The foliation geodesic flow preserves $\hat{\mathcal{F}}$. The Riemannian geometry of this flow has been studied by Walczak in a series of papers [116, 117, 118, 119]. It also has a close relationship to the entropy of foliations.

Ghys, Langevin and Walczak actually defined two entropies for a foliation in [37], the pseudogroup entropy $h(\mathcal{F})$ introduced above, and another invariant called the *geometric entropy* of $\mathcal{F}$, defined by requiring points in an (ϵ, n)-separated set to be separated by the holonomy along geodesic segments of length at most n. Equivalently, the geometric entropy is that for the pseudogroup $\mathcal{G}_{\mathcal{F}}$ with the norm on maps defined via the shortest leafwise distance of a leafwise path with the same holonomy. Thus, as proved in [37], these entropies are either both zero, or non-zero.

The paper [59] proposed to reformulate the geometric entropy of $\mathcal{F}$ in terms of the entropy relative to the invariant foliation $\hat{\mathcal{F}}$ associated to the foliation geodesic flow, and then use the restriction of the normal derivative cocycle $D\varphi\colon \mathcal{G}_{\mathcal{F}} \to \mathrm{GL}(\mathbb{R}^q)$ to this flow to estimate the entropy.

This would allow introduction of many techniques of smooth dynamical systems for the study of $h(\mathcal{F})$, and give a uniform approach which applied in all codimensions along the lines used in [66].

The papers [58, 66] used ergodic theory methods to prove vanishing theorems for secondary classes in terms of the range of the derivative cocycle $D\varphi$ on the metric equivalence relation defined by the pseudogroup $\mathcal{G}_{\mathcal{F}}$. A natural question following these works was to find geometric conditions which gave information about the Lyapunov band spectrum of the transverse derivative cocycle $D\varphi$, and use this information to obtain further relations between dynamics and the secondary classes. For example, the paper [57] showed that the characteristic classes must vanish for distal group actions with some additional hypotheses. A key step in [57] was to introduce invariant measures, not for the full group action, but associated to particular elements of the group along which there was hyperbolic expansion, and consider the Pesin stable manifold for these measures. The corresponding approach for foliations is to consider transversally hyperbolic invariant measures for the geodesic flow, and their stable manifolds. So, the techniques of [57, 58, 66] can be applied to the study of the relation between $h(\mathcal{F})$ and the dynamics of $\mathcal{F}$.

Two difficulties in applying these ideas to the study of foliation geometric entropy are immediately encountered: First, the entropy of the geodesic flow, relative to the invariant foliation, clearly bounds the geometric entropy from below. For foliations whose leaves have subexponential growth, the relative geodesic flow entropy and the geometric entropy are equivalent. Langevin and Walczak [74] called the relative geodesic flow entropy the *transverse entropy*, and proved that in codimension one, the geometric entropy and the transverse entropy are always equivalent. But it is not known whether these two entropies always agree for codimension greater than one.

The other difficulty is the definition and applications of a measure entropy for foliations, based on a relative measure entropy for the geodesic flow analogous to a construction well known in the ergodic theory of maps [78]. The (formidable) technical work required has never been written up, and it remains an open problem to justify this part of the program.

Langevin and Walczak [75] also studied the relations between the geometric entropy and the exponents of the transverse derivative cocycle $D\varphi$. This paper introduces the “pressure” for the dynamics of a pseudogroup. Biś and Walczak [2] showed that the geometric entropy can be calculated using pseudo-orbits.

During 1999-2000, the program of [59] for the study of $h(\mathcal{F})$ was carried out for codimension one foliations, avoiding the construction of relative measure entropy [62]. The role of the relative measure entropy was to obtain lower bound estimates on the geometric entropy in terms of the Lyapunov spectrum of invariant measures for the foliation geodesic flow. The alternative approach of [63, 64, 65] is based on the construction of special dynamical subsystems, called “ping-pong games”. The existence of a ping-pong game is equivalent to the existence of a resilient leaf, and directly implies the geometric entropy of the foliation is positive. The key point is that asymptotic estimates for the restriction of $D\varphi$ along the foliation geodesic flow can be combined with stable manifold theory along transversally hyperbolic invariant measures for the geodesic flow, to produce ping-pong games, and hence prove the foliation geometric entropy must be positive.

The dynamic given by a ping-pong game represents an intermediate invariant of a foliation, and its existence can be proved by techniques similar to proving the existence of measures with positive relative entropy. A good theory of relative measure entropy should assign positive values to the dynamical subsystem determined by a ping-pong game. In the absence of such a theory, the study of the ping-pong games of $\mathcal{F}$ provides an effective method to estimate $h(\mathcal{F})$. Here are the main results from the papers [63, 64, 65]:

THEOREM 5.3. (Theorem 1.1, [63]) *Suppose $\mathcal{F}$ is a codimension one, transversally C^1-foliation with $h(\mathcal{F}) > 0$, then $\mathcal{F}$ has a ping-pong game, and hence a resilient leaf.*

The proof of this uses "elementary" techniques of ergodic theory and flow dynamics, so avoids the use of C^2-structure theory.

THEOREM 5.4. (Theorem 1.1, [65]) *Suppose $\mathcal{F}$ is a codimension one, transversally C^1-foliation with non-trivial Godbillon measure $g_{\mathcal{F}}$, then $\mathcal{F}$ has a ping-pong game and hence $h(\mathcal{F}) > 0$.*

The proof of this uses a new cocycle tempering method, modifying that of [58], and the basic methods for constructing ping-pong games developed in [63, 64].

For higher codimensions, this approach also yields the following result, where it is necessary to impose a Hölder condition on the transverse holonomy:

THEOREM 5.5. (Theorem 1.3, [65]) *If $\mathcal{F}$ is a codimension q, transversally C^{1+a}-foliation with $h(\mathcal{F}) > 0$, then $\mathcal{F}$ is not distal. In particular, $\mathcal{F}$ cannot be a foliation with all leaves compact.*

The proofs of these results are heavily "one dimensional", but it seems likely that it will be possible to extend the techniques of proof to higher codimensions, to obtain relations between $h(\mathcal{F})$ and the secondary classes. (See also the discussion for Problem 9.5 at the end of this paper.)

6. Exceptional minimal sets

A leaf of a codimension one foliation is *semi-proper* if it approaches itself from one side. If the closure $\mathbf{K} = \overline{L}$ of a semi-proper leaf L is minimal, then it is transversally a nowhere dense Cantor set, and we say $\mathbf{K}$ is an *exceptional minimal set.* The semi-proper leaves are precisely the leaves through the endpoints in the gaps of the transverse Cantor set. Semi-proper leaves and exceptional minimal sets have been understood as an essential part of the study of codimension one dynamics since the work of Denjoy, and the generalizations to foliations by Sacksteder [96, 100].

A variety of constructions of exceptional minimal sets have been given (e.g., see [99, 46, 68, 80]). The Poincaré-Bendixson classification theory for C^2-foliations [9, 49] is greatly simplified when there are no exceptional minimal sets. The proof of Duminy's theorem assumes there are no resilient leaves, and hence no exceptional minimal set, and proceeds from there.

Surprisingly, the exceptional minimal sets have defied an easy classification.

One of the open problems is to show that an exceptional minimal set for a C^2-foliation must have Lebesgue measure zero. Inaba [68] and Matsumoto [80] in 1986 gave general constructions of minimal sets for which the Lebesgue measure is zero. As remarked in [80], if an exceptional minimal set $\mathbf{K}$ has measure zero, then the Godbillon-Vey measure vanishes on $\mathbf{K}$, so that analysis of the Godbillon-Vey invariant can ignore the contributions from exceptional minimal sets. Though unlikely, it would be an amazing result to show there exists an exceptional (local) minimal set with non-zero Godbillon-Vey measure.

Another open problem is to show that the leaves in an exceptional minimal set of a C^2-foliation must have a Cantor set of ends. A deep, unpublished work of Duminy (written up by Cantwell and Conlon in the manuscript [17]) shows that the semi-proper leaves always have a Cantor set of ends.

One can also ask when an exceptional minimal set contains only a finite number of semi-proper leaves. There exists C^1-foliations with exceptional minimal sets having a countably infinite number of semi-proper leaves, but it is conjectured that for C^2-foliations this is impossible.

In spite of the extensive study of exceptional minimal sets in codimension one dynamics, all of these questions show a key piece of the puzzle is still missing. To quote from [14], "Our very incomplete understanding of the exceptional type constitutes a major gap in the theory."

In 1988, Cantwell and Conlon [14] introduced a class of exceptional minimal sets, those of *Markov type*. A Markov minimal set **K** admits a finite set of expanding holonomy generators which define the foliation dynamics on it, and thus the dynamics on **K** are given by a quotient of a subshift of finite type. Cantwell and Conlon showed that for an exceptional minimal set of Markov type, there are only a finite number of semi-proper leaves, and **K** has measure zero [14]. They later showed that every leaf in a Markov minimal set **K** has a Cantor set of ends [16].

Inaba and Matsumoto [69] showed for transversally projective foliations, an exceptional minimal set is always Markov. This paper also gave a technical refinement of the definition of Markov property, which broadens the definition. Walczak [120] showed in fact that a Markov minimal set has Hausdorff dimension less than the dimension of M.

7. Extensions of Godbillon-Vey

The definition of the Godbillon-Vey invariant of a C^2-foliation of codimension one clearly depends upon the second derivatives of the 1-form defining $\mathcal{F}$. It is conjectured that if there is a homeomorphism $h\colon M \to M'$ mapping the leaves of a C^2 foliation $\mathcal{F}$ on M to the leaves of a C^2-foliation $\mathcal{F}'$ on M', then $h^*GV(\mathcal{F}') = GV(\mathcal{F})$. This problem has been posed in each of the foliation surveys and problems sessions [102, 31, 72] since 1978, and still remains unresolved.

The existence of a topological conjugacy h between $\mathcal{F}$ and $\mathcal{F}'$ means that the two foliations have the same topological dynamics. If $GV(\mathcal{F})$ is determined by the topological dynamics of $\mathcal{F}$, then the conjecture should be true. On the other hand, conjugation by a homeomorphism allows changing the exponents of hyperbolicity for the dynamics. For example, a homeomorphism can change the "shape" of an exceptional minimal set, and similarly distort the transverse dynamics of a foliation. Thus, it would be very surprising if the conjecture is true. However, there are no counter-examples in C^2, and no suggestive examples to guide intuition on this question.

One approach to the problem would be to show that non-triviality of the Godbillon-Vey classes implies there is sufficient hyperbolicity in the dynamics of the foliations (at least in the support of their Godbillon-Vey measures) to prove that the homeomorphism h is actually C^1 or possibly even C^2, along the lines of [39] mentioned below.

A weaker conjecture is whether the conclusion holds when the conjugacy h has some additional regularity. For example, Raby [94] showed that $GV(\mathcal{F})$ is an invariant under C^1-diffeomorphism. Hurder and Katok [67] showed (independently, and using essentially the same methods as Raby) that if the conjugacy h and its inverse are absolutely continuous, then the conclusion is true. Natsume [87] showed that the Godbillon-Vey map in analytical K-theory of foliation C^*-algebras is also C^1-invariant.

Ghys and Tsuboi [39] used Duminy's Theorem to show that for a foliation with $GV(\mathcal{F}) \neq 0$, a C^1-conjugacy must be C^2 on the support of the Godbillon-Vey measure, thus the C^1-conjecture is only of modest interest. It is unknown if the assumption that h and its inverse are Hölder C^a-continuous, for some $a > 0$, suffices to show $h^*GV(\mathcal{F}') = GV(\mathcal{F})$.

There are counter-examples to topological invariance of extensions of the Godbillon-Vey class to foliations which are not C^2. Ghys defined in [30] a "Godbillon-Vey" type invariant for piecewise C^2-foliations in codimension one, and then showed via surgery on Anosov flows on 3-manifolds that there are homeomorphic, piecewise C^2-foliations with distinct "Godbillon-Vey" invariants. In another direction, Hurder and Katok defined in [67] a "Godbillon-Vey" type invariant for the weak-stable foliations of volume preserving Anosov flows on 3-manifolds, and showed that for the geodesic flow of a metric of variable negative curvature on a compact Riemann surface, the "Godbillon-Vey" invariant is a function of the "Mitsumatsu defect" [81], hence varies continuously and non-trivially as a function of the metric (Corollary 3.12, [67]) . The weak-stable foliations of all of these metrics are topologically conjugate, so this gives a continuous family of counter-examples. Both approaches to extending the Godbillon-Vey invariant are combined in work of Tsuboi [108, 109].

The topological invariance conjecture highlights once again how little is fundamentally known about the Godbillon-Vey class and its geometric or dynamical meaning. Perhaps, if it were possible to give a geometric or dynamic interpretation for the "Vey class", then one could determine whether the ingredients are preserved under homeomorphism, or possibly under a Hölder homeomorphism, or just under diffeomorphism.

8. Tricks and treats

While compiling this survey, and from the author's own research in the subject, several techniques and methods frequently are seen to be both essential and in some way unique to the study of the Godbillon-Vey classes and the dynamics of codimension one foliations. In this section, we compile a short sampling and brief description of selected techniques which have led to a deeper understanding of the themes of this survey. Of course, a thorough reading of the introductory texts on foliations, such as Godbillon's book [40] or the recent text by Candel and Conlon [6], reveals a far greater variety of ideas and techniques than discussed below. Still, it seems useful (and novel) to offer a list of "Tricks and Treats" for the subject, if only as an advertisement for the variety of methods which play a role in this field.

TECHNIQUE 8.1. *Naive distortion lemma*

This is the most well-known method of 1-dimensional dynamics, except perhaps the well-ordering of the line. The hypothesis "$\mathcal{F}$ is C^2" often means simply that the elements of the holonomy pseudogroup satisfy this estimate, which was used in the celebrated theorems of Denjoy [24] and Sacksteder [100]. Briefly, recall that given a chain of local diffeomorphisms $g = h_{i_n} \circ \cdots h_{i_1}$ and two points u_0 and v_0 in the domain of g, then

$$|\log\{g'(u_0)\} - \log\{g'(v_0)\}| \leq \theta \sum_{p=0}^{n-1} |u_p - v_p|$$

where θ is a constant depending on the C^2 norms of the generating elements $\{h_1, \ldots, h_N\}$ and $u_p = h_{i_p} \circ \cdots h_{i_1}(u_0)$ and $v_p = h_{i_p} \circ \cdots h_{i_1}(v_0)$. See section 8.1.A, [6]. This is usually applied in a context where the right-hand-sum is estimated by geometric considerations, as occurs when u_0 and v_0 are the endpoints of a gap for an exceptional minimal set. It is called "naive" by Sullivan, because in the modern theory of 1-dimensional dynamics there is also a "sophisticated" distortion lemma, also known as the "Schwartzian distortion lemma". This latter technique is a fundamental tool for renormalization theory of maps in 1-dimensional dynamics, but has not been used in the study of foliations, though it might prove useful for studying exceptional minimal sets.

TECHNIQUE 8.2. *Octopus decomposition*

This result for C^0-foliations, which has no counterpart in classic dynamics, was introduced by Dippolito [25]. It states that an open saturated subset $U \subset M$ of a foliated compact manifold can be decomposed into a union of manifold with corners, $U = \mathcal{N} \cup A_1 \cup \cdots \cup A_n$ where $\mathcal{N}$ is the body, or nucleus, of U, and the A_i are the arms. The body $\mathcal{N}$ is a compact, connected manifold with boundary and corners, and $\mathcal{F}|\mathcal{N}$ foliates $\mathcal{N}$. Each arm A_i is a closed, non-compact submanifold with boundary and corners, and $\mathcal{F}|A_i$ is a foliated interval bundle. More technical conditions are required (see pages 130-131, [6]) but this suffices to give the idea. The power of this result lies in that the structure of foliated interval bundles is well understood, so this isolates difficulties with the study of $\mathcal{F}|U$ to the compact foliated body $\mathcal{F}|\mathcal{N}$. One imagines the arms of the octopus snaking through the various exceptional minimal sets of $\mathcal{F}$, or possibly squeezing between the proper leaves. The conjecture about whether an exceptional minimal set $\mathbf{K}$ for a C^2 foliation can have infinitely many semi-proper leaves is just asking how many octopi can share the set $\mathbf{K}$!

TECHNIQUE 8.3. *Poincaré-Bendixson theory*

The theory of levels is one of the most sophisticated tools for the study of C^2-foliations. It was developed by Cantwell and Conlon [9] and Hector [49], and defines inductively a decomposition (or hierarchy) of a foliation according to the asymptotics of the leaves. Starting with level 0, the compact minimal sets, the leaves at the next level are the local minimal sets for the complement of the previous level. The resulting structure and complications can be formidable. Fortunately, there is an introduction to the theory [15], and Chapter 8 of [6] provides a detailed and patient discussion of all aspects of the theory. One its greatest successes is the structure theory of real analytic foliations [11], where the hierarchy is finite and the structure of each stage is very well understood. When this theory is applied to codimension one foliations of 3-manifolds, one can ask about the "placement" of the leaves, or local minimal sets at various levels, within the manifold. For example, if M is Haken it is possible to formulate precise questions about the leaf placements at increasing levels and the fundamental group of M. Unfortunately, it is not possible to answer these questions yet – the ongoing work of Cantwell and Conlon [18, 19, 20, 21] have solved the placement problem for depth one, while depth two awaits.

TECHNIQUE 8.4. *Micro-expansion, sheaves and quivers*

A C^1-foliation $\mathcal{F}$ with positive geometric entropy has micro-expansion, a phrase coined to suggest the explosion in the orbits of exponentially close points that has to occur when $h(\mathcal{F}) > 0$. A complete transversal $\mathcal{T}$ for $\mathcal{F}$ has finite length, so by definition of geometric entropy, for $\epsilon > 0$ small and $i \to \infty$, there exists a sequence $n_i \to \infty$ and collections of (ϵ, n_i)-separated points $\{x^i_1, \ldots, x^i_{p_i}\} \subset \mathcal{T}$ where the sequence $\{p_i\}$ has exponential growth $\exp(n_i h(\mathcal{F}))/p_i \to 1$. By a pigeon-hole principle, there must be subcollections of exponentially many points exponentially close which are (ϵ, n_i)-separated. We can assume the set transversal $\mathcal{T}$ is a subset of the line so well-ordered, and the sets are index with $x^i_k < x^i_\ell$ for $k < \ell$. Then for each x^i_k there is an element of holonomy γ^i_k of length at most n_i which ϵ-separates x^i_k and x^i_{k+1}. One pictures this as a *sheaf* of arrows of length n_i with bases concentrated on smaller and smaller regions, the arrows representing elements of holonomy $\{\gamma^i_k\}$, and the tips pairwise separating so they end up at least ϵ apart. Moreover, the transverse derivatives along these arrows are non-uniformly hyperbolic by the mean value theorem, so these could be called *hyperbolic quivers*, which sounds even more formidable.

On the other hand, the tips of the arrows must also land somewhere on $\mathcal{T}$, and since there are exponentially many arrows, another application of the pigeon-hole principle yields exponential families of arrows (elements of holonomy) of length n_i whose bases and tips are exponentially close. These are *quivers*. Thus, there is quite a lot of dynamical information contained in the statement "$h(\mathcal{F}) > 0$". As intuitive and imprecise as the image of quivers may seem, it can be formalized in a variety of contexts to help understand the geometric entropy. For example, it is a key idea behind the proofs involving entropy in the papers [63, 64, 65]. The main question about quivers is how to quantify them. It seems likely that any scheme will be analogous to the construction of relative measure entropy for the geodesic flow of $\mathcal{F}$. There should also be a connection between quivers and the pseudo-orbit estimates of [2].

Almost the opposite concept was used in the proof of Theorem 5.1, [37] which proved that if $h(\mathcal{F}) = 0$ then there exists a holonomy invariant transverse measure. Here, the intuition is closer to a "straw mat", as the base of the arrows are uniformly spaced and the tips cluster at a subexponential rate.

TECHNIQUE 8.5. *Ping-pong games and closing*

A ping-pong game for a C^0-foliation is a basic concept of topological dynamics, which is equivalent to the idea of resilient orbits, but somehow more natural to construct. The basic idea is that there should be given an interval $I_0 \subset \mathcal{T}$ in the transversal for $\mathcal{F}$, and elements of holonomy $h_1 \colon I_0 \to I_1 \subset I_0$ and $h_2 \colon I_0 \to I_2 \subset I_0$ so that both h_1 and h_2 are contractions. We also require that the closures of I_1 and I_2 are disjoint, and both contained in the interior of I_0. It is an exercise that the orbits of the fixed-points in a ping-pong game generate resilient leaves, and that a foliation with a ping-pong game must have $h(\mathcal{F}) > 0$. This latter fact was used in [37] to show that a foliation with a resilient leaf has $h(\mathcal{F}) > 0$. For a C^1-foliation, a hyperbolic ping-pong game is one where the maps are hyperbolic contractions on the set I_0, and thus h_1 and h_2 must each have a unique hyperbolic fixed-point. Thus, a hyperbolic ping-pong game is like having part of the hypothesis of a Markov minimal set, except that orbits of the hyperbolic fixed-points are not assumed to lie in an exceptional minimal set. A C^1-foliation with a quiver must have a ping-pong game, and hence positive entropy – a fact used in the proof of Theorem 1.1 of [65]. The existence of a quiver also implies there are attracting fixed-points, which is a type of closing lemma that says close to every quiver is a hyperbolic fixed-point. This idea is developed in detail in section 4 of [64].

TECHNIQUE 8.6. *Tempering procedures*

Tempering procedures are specialty tools, which often appear to be more technical than useful, but when needed are indispensable and very powerful. Tempering is a process which converts asymptotic estimates for a cocycle into local estimates by making a change of scale (coboundary) for the initial data. For example, if $\varphi \colon \Gamma \times \mathbb{S}^1 \to \mathbb{S}^1$ is a C^1-action on the circle, then the derivative $d\varphi \colon \Gamma \times \mathbb{S}^1 \to \mathbb{R}$ is a real-valued cocycle over the action φ. We define the exponent of $d\phi$ at x by

$$\lambda(x) = \limsup_{n\to\infty} \frac{\log\left(\max\{\varphi(\gamma)'(x) \mid \|\gamma\| \leq n\}\right)}{n}$$

If $\mathbf{K} \subset M$ is a closed saturated subset such that $\lambda(x) \leq a$ for almost all $x \in \mathbf{K}$, then for $\mathrm{gr}(\mathcal{F})$ the word growth rate of Γ, choose $\epsilon > a + \mathrm{gr}(\Gamma)$ and define

$$f_\epsilon(x) = \sum_{\gamma\in\Gamma} \exp\{-\epsilon \cdot \|\gamma\|\} \cdot d\phi(\gamma, x)$$

It is then an exercise that the new cocycle $\tilde{\phi}(\gamma, x) = f_\epsilon(\varphi(\gamma)(x)) \cdot \phi(\gamma, x) \cdot f_\epsilon(x)^{-1}$ has uniform local variation bounded by ϵ.

Tempering for linear cocycles over group actions and foliations was introduced by Hurder and Katok in [67], and used there and in [58] to regularize the transverse derivative cocycle $D\varphi$ for foliations. A discussion of tempering can also be found in [31]. For example, when the leaves of $\mathcal{F}$ have subexponential volume growth, the transverse Radon-Nikodyn cocycle satisfies $\lambda(x) = 0$ for almost all x, so tempering constructs a transverse measure defining $\mathcal{F}$ with arbitrarily small local variation. The papers [64, 65] introduce another tempering procedure for subexponential growth cocycles over arbitrary growth foliations, which is applied in [65] to obtain new vanishing theorems for the Godbillon-Vey classes. Tempering procedures are essentially the only available method for converting transversally measurable information for a foliation (typically obtained from ergodic theory considerations) into differential geometric conclusions.

9. Open questions

"Problem sessions" were held at both the 1976 and 1992 symposia on foliations at Rio de Janeiro, with the discussion and proposed problems compiled by Paul Schweitzer [102] for the 1976 meeting, and Remi Langevin [72] for the 1992 meeting. Some of the problems remain unchanged, while a comparison between these two reports fourteen years apart illustrates some of the advances in the field, and changing emphasis in research. The survey of the Godbillon-Vey invariant by Ghys [31] also includes a number of problems with discussions about them. Here, we compile a list of questions concerning the topics of the present survey. It is not meant to be comprehensive when compared to the more general problem lists above, but does attempt to include all of the frequently mentioned problems regarding the Godbillon-Vey classes and foliation dynamics.

Problems on Godbillon-Vey Invariants

PROBLEM 9.1. *Give a* geometric *interpretation of the Godbillon-Vey invariant*

The Moussu-Pelletier and Sullivan Conjecture is a one-sided look at $GV(\mathcal{F})$, as it only relates to dynamical properties of $\mathcal{F}$ which can force the Godbillon measure to vanish. The other side is the "Vey class" which depends upon curvature properties of the leaves and normal bundle. The Reinhart-Wood formula [98] gave a *pointwise* geometric interpretation of $GV(\mathcal{F})$ for 3-manifolds. What is needed is a more global geometric property of $\mathcal{F}$ which is measured by $GV(\mathcal{F})$. The helical wobble description by Thurston [106] is a first attempt at such a result, and the Reinhart-Wood formula suitably interprets this idea locally. Langevin has suggested that possibly the Godbillon-Vey invariant can be interpreted in the context of integral geometry and conformal invariants [3, 73] as a measure in some suitable sense. The goal for any such an interpretation, is that it should provide *sufficient* conditions for $GV(\mathcal{F}) \neq 0$.

PROBLEM 9.2. *Topological invariance of the Godbillon-Vey invariant*

Given a homeomorphism $h\colon M \to M'$ mapping the leaves of a C^2 foliation $\mathcal{F}$ on M to the leaves of a C^2-foliation $\mathcal{F}'$ on M', show $h^*GV(\mathcal{F}') = GV(\mathcal{F})$. As discussed in section 7, if h is C^1, then Raby [94] proved $h^*GV(\mathcal{F}') = GV(\mathcal{F})$, and when h and its inverse are absolutely continuous, then Hurder and Katok [67] showed this. An intermediate test case might be to assume h and its inverse are a Hölder C^a-continuous for some $a > 0$, and then prove $h^*GV(\mathcal{F}') = GV(\mathcal{F})$, using for example arguments from regularity theory of hyperbolic systems and an approach similar to Ghys and Tsuboi [39]. Alternately, a direct proof may be possible, perhaps based on a solution to Problem 9.1.

PROBLEM 9.3. *The Godbillon-Vey invariant and harmonic measures*

The vanishing theorems are based on relating the Godbillon measure to the existence of "almost invariant" smooth transverse measures for $\mathcal{F}$. A foliation always admits a harmonic measure, but the structure of that measure depends upon whether $\mathcal{F}$ admits transverse invariant measures, or not. Is it possible to establish relations between the values of the Godbillon-Vey invariant and the structure of harmonic measures for $\mathcal{F}$? There are other similarities in the properties of both of these invariants of $\mathcal{F}$ which suggests that such a relationship is plausible.

PROBLEM 9.4. *What is meaning of thickness?*

The concept of "thickness" introduced by Duminy [26, 27, 12] was given in terms of the structure theory of C^2-foliations, yet its application is to show the foliation admits almost invariant transverse volume forms on an open saturated subset, which is a purely dynamical consideration. Does the thickness have an interpretation as a dynamical property of the foliation geodesic flow, or some other ergodic property of $\mathcal{F}$?

PROBLEM 9.5. *Suppose that $\mathcal{F}$ has codimension $q > 1$ and there is some non-zero secondary class (or possibly Weil measure). Does this imply $h(\mathcal{F}) > 0$?*

Hurder [56] showed that for a C^2-foliation of codimension $q > 1$, if there is a leaf L whose linear holonomy map $D\varphi\colon \pi_1(L, x) \to \mathrm{GL}(\mathbb{R}^q)$ has non-amenable image, then $\mathcal{F}$ has leaves of exponential growth. The proof actually constructs a modified ping-pong game for $\mathcal{F}$, using the C^2-hypothesis to show that the orbits of the holonomy pseudogroup shadow the orbits of the linear holonomy group which has an actual ping-pong game by Tits [107]. Thus, it seems probable that this proof also shows $h(\mathcal{F}) > 0$ with these hypotheses. Since the Weil measures vanish for a foliation whose transverse derivative cocycle $D\varphi\colon \mathcal{G}_{\mathcal{F}} \to \mathrm{GL}(\mathbb{R}^q)$ has amenable algebraic hull [66], it should be possible to combine the methods of [56, 66, 63] to solve this problem.

Problems on Minimal Sets

PROBLEM 9.6. *Let* $\mathbf{K}$ *be an exceptional minimal set for a codimension one C^2-foliation $\mathcal{F}$. Show*

1. *The Lebesgue measure of* $\mathbf{K}$ *is zero;*
2. $\mathbf{K}$ *has only a finite number of semi-proper leaves;*
3. *Every leaf of* $\mathbf{K}$ *has a Cantor set of ends.*
4. *Every semiproper leaf of* $\mathbf{K}$ *has germinal holonomy infinite cyclic, generated by a contraction.*
5. $\mathbf{K}$ *is Markov.*

The first four questions were posed at least 20 years ago. Note that $\mathbf{K}$ has only a finite number of semiproper ends if and only if its complement in M has only a finite number of connected open components. Duminy's Theorem [17] shows that the semiproper leaves of $\mathbf{K}$ must have a Cantor set of ends. A number of authors have shown the measure of $\mathbf{K}$ is zero for special cases [68, 80, 69]. Cantwell and Conlon showed that if $\mathbf{K}$ is Markov, then the first four properties follow [14, 16].

PROBLEM 9.7. *For a codimension one C^2-foliation $\mathcal{F}$, give an example of an exceptional minimal set* $\mathbf{K}$ *with non-trivial Godbillon-Vey measure.*

This is most likely impossible, as it contradicts (9.6.1) above. If a counter-example to (9.6.1) can be constructed, then it will automatically have non-zero Godbillon measure, as an exceptional minimal set must be hyperbolic for a C^2-foliation, so it would then be plausible to ask that whether the Godbillon-Vey class localized to $\mathbf{K}$ is non-zero.

PROBLEM 9.8. *For a codimension one, C^1- or C^2-foliation $\mathcal{F}$, give a structure theorem for the exceptional minimal sets of $\mathcal{F}$.*

This is asking first for an understanding of how many semiproper leaves there are, and then for some sort of generalized Markov structure on $\mathbf{K}$. In other words, it is asking a lot!

PROBLEM 9.9. *Let* Γ *be a* closed *subgroup of* $\mathrm{Homeo}_+(\mathbb{S}^1)$ *with a dense orbit. Is* Γ *conjugate to one of the subgroups* $\mathrm{SO}(\mathbb{R}^2)$*,* $\mathrm{PSL_k}(\mathbb{R}^2)$*, or* $\mathrm{Homeo_{k,+}}(\mathbb{S}^1)$ *of* $\mathrm{Homeo}_+(\mathbb{S}^1)$*?*

This question was posed by Ghys as Problem 4.4, [35]. The hypothesis the group is closed is essential, so unless the group is finite it must be non-discrete. The problem is included as an understanding of this question would surely help with understanding the minimal actions of countable groups on $\mathbb{S}^1$. (Note that the subscript k in the question indicates the k-fold covering group.)

Problems on Geometric Entropy

PROBLEM 9.10. *Give a definition of the* measure *entropy, or some other entropy-type invariant, of a* C^1*-foliation* $\mathcal{F}$*, which can be used to establish positive lower bounds for the geometric entropy.*

This problem was asked in the original paper of Ghys, Langevin and Walczak [37]. Their earlier paper [36] gave a possible definition, but the connection to the geometric entropy is unclear. The paper by Hurder [59] proposes a definition of the measure entropy in terms of invariant measures for the associated geodesic flow. If well-defined, these measure entropies will estimate the entropy of the geodesic flow relative to the invariant foliation almost by definition. A special case is to show there exists a good definition of measure entropies for codimension 1 foliations. Another approach might be to define measure entropy for a foliation in terms of its harmonic measures.

PROBLEM 9.11. *Is positive geometric entropy a generic property for* C^1*-foliations?*

Given any foliation, is there a C^1-close foliation whose entropy is positive?

Problems on Ergodic Theory

PROBLEM 9.12. *Show the set of leaves with non-exponential growth, and not subexponential growth, has Lebesgue measure zero.*

Hector's construction in [47] of examples with leaves of this special type appear to produce a set (of such leaves) with measure zero. This growth condition, that $\limsup \neq \liminf$, implies a high degree of non-uniformity for the asymptotics of the leaf. If there exists a set of positive measure consisting of such leaves, then recurrence within the set should imply a uniformity of the growth, contradicting the hypothesis.

PROBLEM 9.13. *Can a codimension one foliation have higher rank?*

The celebrated theorems of Burger and Monod [5] and Ghys [34] show that a higher rank group does not admit an effective C^1-action on the circle. One can view these results as about the holonomy groups of a codimension one foliation transverse to a circle bundle. Can these theorems be generalized to codimension one foliations which are not transverse to a circle bundle? Part of the problem is to give a suitable definition of higher rank for a foliation (cf. Zimmer [123].)

PROBLEM 9.14. *Does restricted orbit equivalence preserve geometric entropy?*

Given foliated compact manifolds $(M, \mathcal{F})$ and $(M', \mathcal{F}')$, a *restricted orbit equivalence* between $\mathcal{F}$ and $\mathcal{F}'$ is a measurable isomorphism $h \colon M \to M'$ which maps the leaves of $\mathcal{F}$ to the leaves of $\mathcal{F}'$, and the restriction of h to leaves is a coarse isometry for the leaf metrics. Note that h and its inverse are assumed to preserve the Lebesgue measure class, but need not preserve the Riemannian measure. Such a map preserves the Mackey range of the Radon-Nikodyn cocycle [122]. Restricted orbit equivalence also preserves the entropy positive condition, for ergodic $\mathbb{Z}^n$ actions. Does a corresponding result hold for geometric entropy: if $h(\mathcal{F}) > 0$, must $h(\mathcal{F}') > 0$ also?

PROBLEM 9.15. *How is the flow of weights for $\mathcal{F}$ related to the dynamics of $\mathcal{F}$?*

Connes has show that the Godbillon-Vey class, or more precisely the Bott-Thurston 2-cocycle defined by it, can be calculated from the flow of weights for the von neumann algebra $\mathfrak{M}(M, \mathcal{F})$ (see [22], Chapter III.6, [23].) This gives another proof of the theorem of Hurder and Katok [66] that if $GV(\mathcal{F}) \neq 0$ then $\mathfrak{M}(M, \mathcal{F})$ has a factor of type III. The flow of weights is determined by the flow on the Mackey range of the modular cocycle, so that a type III factor corresponds to a ergodic component of the Mackey range with no invariant measure. However, almost nothing else is known about how the flow of weights is related to the topological dynamics of $\mathcal{F}$. In particular, Alberto Candel has asked whether the existence of a resilient leaf can be proven using properties of the flow of weights.

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Steven Hurder
Department of Mathematics (m/c 249)
University of Illinois at Chicago
851 S. Morgan St.
CHICAGO, IL 60607-7045 USA
Email: hurderuic.edu
Web: http://www.math.uic.edu/∼hurder/